

Reachability in succinct one-counter games[☆]

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Abstract

We consider the reachability problem on transition systems corresponding to succinct one-counter machines, that is, machines where the counter is incremented or decremented by a value given in binary.

1. Preliminaries

We are interested in reachability problems on transition graphs defined by one-counter machines where:

- The counter may take any integer value (including negative values);
- The counter is incremented or decremented by binary weights;
- Additional transitions are available to the machine if the counter does or does not have value 0.

Previous work [1, 2] has considered other variations on these initial assumptions.

Formally, a transition graph of a one-counter machine (or one-counter graph) is given by a tuple $(V, V_{\exists}, E, E_0, E_{\neq 0}, q_0, w)$ where V is a finite set of *states*; $V_{\exists} \subseteq V$ are the *states of Eve* ($V \setminus V_{\exists}$ are the *states of Adam*); $E, E_0, E_{\neq 0} \subseteq V \times V$ [E_0 ($E_{\neq 0}$) is the set of edges (*de*)*activated at 0*]; $q_0 \in V$ is the *initial state*; and $w : E \rightarrow \mathbb{Z}$ is the *weight function*. The (infinite) unweighted arena defined by such a tuple has:

- Vertex set: $V \times \mathbb{Z}$,
- Eve's vertices: $V_{\exists} \times \mathbb{Z}$,
- Initial vertex: $(q_0, 0)$,
- For every $e = (v, v') \in E$ and $c \in \mathbb{Z}$ an edge from (v, c) to $(v', c + w(e))$,
- For every $e = (v, v') \in E_0$ an edge from $(v, 0)$ to $(v', 0)$, and
- For every $e = (v, v') \in E_{\neq 0}$ an edge from (v, c) to (v', c) for $c \neq 0$.

1.1. Reachability problems

We are interested in the following reachability problems listed in increasing order of difficulty. They are all known to be in EXPSPACE and EXPTIME-hard. Finite memory strategies suffice for all but parity games.

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Global reachability. Given $t \in \mathbb{Z}$ does Eve have a strategy to get the counter to value t (in any state)? That is, can she force the play to (v, t) for some $v \in V$? EXPTIME-hard via straightforward reduction from countdown games.

Reachability. Given $t \in \mathbb{Z}$ and $F \subseteq V$ does Eve have a strategy to get the counter to value t in a state of F ?

Büchi (repeated reachability). Given $F \subseteq V$ does Eve have a strategy to infinitely often have the counter with value 0 whilst in a state of F ?

Parity. Given a priority function $\Omega : V \rightarrow \mathbb{N}$ which defines a priority function on the infinite arena in the obvious way, does Eve win the (infinite) parity game? Known to be in EXPSPACE by the result in [3] which gave a PSPACE algorithm for parity games on unary-encoded one-counter graphs.

2. EXPSPACE-completeness of succinct one-counter games

We will prove the following:

Theorem 1. *Determining if Eve has a winning strategy in any of these games is EXPSPACE-complete.*

From the above results it suffices to show EXPSPACE-hardness.

2.1. Simplifying assumptions

Activating/Deactivating edges. It might seem that including edges that are (de)activated when the counter is 0 might yield a more powerful model, but we can use the antagonistic nature of the game to simulate (de)activating edges. That is, we activate all transitions but give the other player the ability to punish the (non-)zeroness of the counter. Figures 1 and 2 show the gadgets that simulate an activating edge (v, v') , and Figures 3 and 4 show gadgets that simulate a deactivating edge (v, v') . In all figures unlabelled edges have weight 0, square nodes represent states owned by Eve, circle nodes represent states owned by Adam, and all sinks are included in the target set.

Target set. We can assume that $F \subseteq V_{\exists}$ as follows: for every $v \in F \setminus V_{\exists}$ we add a new vertex $v' \in V_{\exists} \cap F$ and edge (v', v) [with weight 0] and replace all edges (u, v) with (u, v') [with the same weight].

2.2. EXPSPACE-hardness of Büchi games

It is well known that CTL model checking (on a transition system) reduces to a two-player game with a Büchi winning condition [4]. The same reduction shows that CTL model checking on succinct one-counter automata reduces to Büchi games on one-counter graphs. In [5], CTL model checking on succinct one-counter automata was shown to be EXPSPACE-complete, hence one-counter games with a Büchi winning condition are also EXPSPACE-hard.

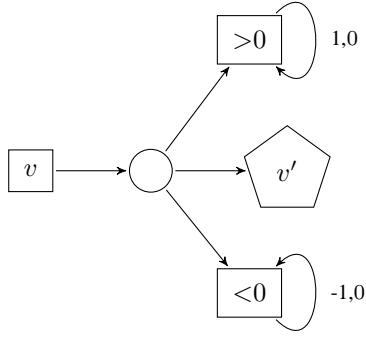


Figure 1: Simulating an activating edge from an Eve state

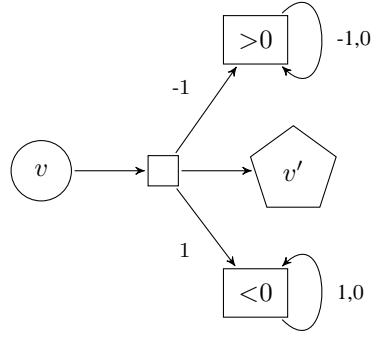


Figure 2: Simulating an activating edge from an Adam state

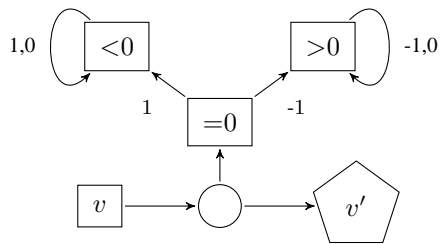


Figure 3: Simulating a deactivating edge from an Eve state

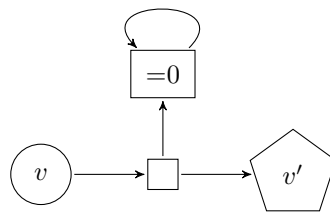


Figure 4: Simulating a deactivating edge from an Adam state

2.3. From Büchi games to Reachability

The following lemma, which is readily established using pumping techniques will be useful.

Given a Büchi game $G = (V, E, E_0, E_{\neq 0}, q_0, F)$ with target set $F \subseteq V_{\exists}$, we construct a new one-counter reachability game as follows:

- The game graph consists of $|F| + 1$ copies of G with a 0-activated edge from (v, i) to $(v, i + 1)$ for all $v \in F$ and $1 \leq i \leq |F|$,
- The initial state is $(q_0, 1)$,
- The target set is $\{(v, |F| + 1) : v \in F\}$, and
- The target value is 0.

Clearly Eve wins this game if and only if in the original game she can reach F with counter value 0 $|F| + 1$ times. Hence if she wins the Büchi game she has a winning in the reachability game. We now show the converse, that is if she can reach F $|F| + 1$ times then she can reach some vertex in F with counter value 0 infinitely often. More precisely we will show how to defeat any positional (w.r.t. the current state and counter value) strategy for Adam in the original Büchi game. It is well known [6] that such strategies are sufficient for winning strategies, thus this is sufficient for our result. Such a strategy has a natural interpretation in the reachability game, so Eve has a counter-strategy to ensure F is visited with counter value 0 $|F| + 1$ times against this strategy. By the pigeon-hole principle there is some vertex $v \in F$ visited at least twice in the play. Hence Eve has a strategy (against Adam's strategy) to reach v with counter value 0 from both q_0 and v . Hence Eve can visit v with counter value 0 infinitely often in the original game.

2.4. From Reachability to Global Reachability

Given a reachability game G with target set $F \subseteq V_{\exists}$ and $E_0 = E_{\neq 0} = \emptyset$, we construct a new arena as follows:

- Double the weights of the edges in G ;
- Add a new (initial) vertex v_0 and a new sink (with 0-weighted loops) v_f ;
- Add an edge of weight +1 from v_0 to the original initial vertex;
- Add edges of weight -1 from F to v_f .

Due to parity arguments the counter can only have value 0 at v_f . Clearly v_f can be reached with value 0 if and only if the target set F can be reached with value 0 in the original game. Thus Eve wins the Global Reachability game on this new arena if and only if she has a winning strategy in the original Reachability game. Thus this gives a reduction from Reachability to Global Reachability. Note that this does not work in the unary case as we utilize the “long-reach” ability of doubled weight values to avoid counter values of 0 in the original game.

3. Super-exponential counter values

For one-counter machines without alternation (i.e. one player games) it is known [7, Lemma 42] that the reachability problem can be solved without the counter value exceeding an exponential bound. Our results show that such a bound in the case of alternating machines is unlikely – it would yield an alternating PSPACE (i.e. EXPTIME) algorithm, thereby implying $\text{EXPTIME} = \text{EXPSPACE}$. We now give a concrete example that shows super-exponential counter values are in fact necessary in succinct one-counter games.

Game summary. The game G_n proceeds as follows:

1. Eve increments the counter to a multiple of 2^n , $M \cdot 2^n$,
2. Adam chooses some odd $m \in (0, 2^n)$ and adds it to the counter,
3. Eve removes a multiple (at least one) of $m \cdot 2^n$ from the counter, and
4. Eve removes some $m' \in (0, 2^n)$ from the counter.

Implementation. Step 1 is implemented by a single Eve vertex with a loop of weight 2^n . Steps 2 and 4 can be implemented by a sequence of n nodes (belonging to the relevant player) where two edges from the i -th to $(i + 1)$ -th vertex allow the relevant player to choose the i -th bit. Step 3 is implemented by a sub-game of n rounds repeated as often as Eve chooses (but at least once). In the i -th round of the subgame Adam chooses the i -th bit b of m . If he chooses correctly then $b \cdot 2^{i+n}$ is subtracted from the counter and the subgame continues, if he chooses incorrectly then the i -th bit is cleared, Eve exits the subgame and clears all but the i -th bit. Note that if Eve tries to exit when Adam chose correctly then the i -th bit is never cleared so Eve is unable to reach a counter value of 0.

Correctness. Clearly Eve can reach a counter value of 0 at the end if and only if $m|M$, $M > 0$ and $m' = m$. Thus in order to win the game M must be a non-zero multiple of all odd numbers in $(0, 2^n)$, in particular it is at least the product of all (odd) primes less than 2^n . Hence $M \geq 2^{\pi(2^n)-1}$, where $\pi(x)$ is the number of primes less than x . Using the standard lower bound of $\frac{x}{\ln x}$ for $\pi(x)$ [8], we have $M \geq 2^{2^n/n-1}$, and hence the counter necessarily attains super-exponential values.

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